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Deformed Boson Scheme including Conventional q-Deformation in Time-Dependent Variational Method. III

—— *Deformation of the su(2,1)-Algebra in Terms of Three Kinds of Boson Operators* ——

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Abstract

Basic idea presented in Parts (I) and (II) for the deformed boson scheme is applied to the case of the $su(2,1)$-algebra for describing many-body systems consisting of three kinds of boson operators. A possible form of the coherent state for the $su(2,1)$-algebra is generalized and the deformed $su(2,1)$-algebra is presented. As an application, the time-evolution of the statistically mixed state in a certain boson system, which has been already investigated by the present authors, is reinvestigated.

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§1. Introduction

In Part (I),[1] we developed a possible form of the deformed boson scheme which is suitable for treating many-body systems constituted of one kind of boson operator. The basic viewpoint can be found in the idea for a generalization of the boson coherent state. In this idea, a certain function of the boson number is introduced and we showed that the deformed boson scheme depends on the choice of the form of this function. In Part (II),[2] we investigated the deformed boson scheme for the system consisting of two kinds of boson operators. Naturally, the idea is a straightforward extension from that in (I). For the system in two kinds of boson operators, the $su(2)$- and the $su(1,1)$-algebra presented by Schwinger[3] are well known. The algebras have been investigated by various authors in terms of the q-deformed algebras.[4] We investigated this case by extending conventional boson coherent states to the forms which are suitable for the deformation of these algebras.

Main aim of Part (III) is to formulate the deformation of the $su(2,1)$-algebra, the generators of which are expressed in terms of the bilinear forms for three kinds of boson operators. Naturally, the basic idea is the same as that adopted in (I) and (II). The present authors already formulated the $su(2,1)$-algebra in the above-mentioned case in Ref.5), which is, hereafter, referred to as (A). Further, the present authors investigated a boson system interacting with an external harmonic oscillator as a possible description of statistically mixed state.[6] Hereafter, Ref.6) is referred to as (B). The basic viewpoint adopted in (B) is found in the idea that the $su(2,1)$-algebra presented in (A) is deformed in the present sense. Then, it may be interesting to reinvestigate the deformation scheme given in (B) under the scheme extended from that adopted in (II).

As was stressed in (II), our basic idea for the deformed algebra is the use of the coherent state. In the case of the $su(2)$- and the $su(1,1)$-algebra, the coherent state is obtained in the form of exponential type of the superposition of the raising operator constructed on the state which vanishes under the operation of the lowering operator, i.e., the minimum weight state. In (III), we adopt the same idea as the above. However, there exists an essential difference between the forms in (II) and (III). In the case of the $su(2)$- and the $su(1,1)$-algebra, there exists only one which plays a role of the raising operator. However, in the case of the $su(2,1)$-algebra in the present form, there exist three operators which can be the candidate of the raising operator. But, for the construction of the coherent state, only two are necessary and, then, the coherent state depends on the choice of the two raising operators. In the present, we adopt the form which plays a central role in (B) and the deformation is performed for this coherent state. As an example of the application, the case discussed in (B) is reinvestigated and, as a natural consequence, we can describe the time-evolution of the statistically mixed state with the same results as those obtained in (B).

In the next section, the $su(2,1)$-algebra in terms of three kinds of boson operators and its classical counterpart are recapitulated. Details have been given in (A). Section 3 is devoted to discussing the deformation of the coherent state, which plays a central role in §2. In §4, the $su(2,1)_q$-algebra is formulated in a certain special case which may be suitable for describing the time-evolution of the statistically mixed state of a boson system with an external harmonic oscillator. Finally, in §5, the formalism in (B) is reformulated in the framework of the present deformed boson scheme. Appendix is devoted to giving some mathematical formulae with the proof.

§2. The $su(2,1)$-algebra in terms of three kinds of boson operators and its classical counterpart

As the preliminary for main parts in the present paper, first of all, we will recapitulate the $su(2,1)$-algebra in a special case : eight generators are expressed in terms of three kinds of boson operators, $(\hat{a},\hat{a}^*)$, $(\hat{b}_1,\hat{b}_1^*)$ and $(\hat{b}_2,\hat{b}_2^*)$. Its details can be seen in (A), in which $(\hat{b}_1,\hat{b}_1^*)$ and $(\hat{b}_2,\hat{b}_2^*)$ should read $(\hat{c},\hat{c}^*)$ and $(\hat{d},\hat{d}^*)$, respectively. The eight generators are expressed as bi-linear functions for the above boson operators in the following form :

$$\hat{T}_+^0=\hat{a}^*\hat{b}_1^* \ , \qquad \hat{T}_-^0=\hat{b}_1\hat{a} \ , \qquad \hat{T}_0=(\hat{a}^*\hat{a}+\hat{b}_1\hat{b}_1^*)/2 \ , \tag{2·1a}$$
$$\hat{S}_+^0=\hat{b}_1^*\hat{b}_2 \ , \qquad \hat{S}_-^0=\hat{b}_2^*\hat{b}_1 \ , \qquad \hat{S}_0=(\hat{b}_1^*\hat{b}_1-\hat{b}_2^*\hat{b}_2)/2 \ , \tag{2·1b}$$
$$\hat{R}_+=\hat{a}^*\hat{b}_2^* \ , \qquad \hat{R}_-=\hat{b}_2\hat{a} \ . \tag{2·1c}$$

The commutation relations for the above generators are given in the form

$$[\hat{T}_+^0,\hat{T}_-^0]=-2\hat{T}_0 \ , \qquad [\hat{S}_+^0,\hat{S}_-^0]=2\hat{S}_0 \ , \tag{2·2a}$$
$$[\hat{T}_\pm^0,\hat{S}_\mp^0]=\mp\hat{R}_\pm \ , \qquad [\hat{T}_\pm^0,\hat{S}_\pm^0]=0 \ , \tag{2·2b}$$
$$\begin{aligned}
&[\hat{T}_0,\hat{T}_\pm^0]=\pm\hat{T}_\pm^0 \ , \qquad [\hat{S}_0,\hat{T}_\pm^0]=\pm(1/2)\cdot\hat{T}_\pm^0 \ ,\\
&[\hat{T}_0,\hat{S}_\pm^0]=\pm(1/2)\cdot\hat{S}_\pm^0 \ , \qquad [\hat{S}_0,\hat{S}_\pm^0]=\pm\hat{S}_\pm^0 \ ,\\
&[\hat{T}_0,\hat{R}_\pm]=\pm(1/2)\cdot\hat{R}_\pm \ , \qquad [\hat{S}_0,\hat{R}_\pm]=\mp(1/2)\cdot\hat{R}_\pm \ ,\\
&[\hat{T}_0,\hat{S}_0]=0 \ ,\\
&[\hat{R}_+,\hat{R}_-]=-2\hat{R}_0 \ , \qquad (\ \hat{R}_0=\hat{T}_0-\hat{S}_0\ ) \ ,\\
&[\hat{R}_\pm,\hat{T}_\pm^0]=0 \ , \qquad [\hat{R}_\pm,\hat{S}_\pm^0]=\mp\hat{T}_\pm^0 \ ,\\
&[\hat{R}_\pm,\hat{T}_\mp^0]=\mp\hat{S}_\mp^0 \ , \qquad [\hat{R}_\pm,\hat{S}_\mp^0]=0 \ .
\end{aligned} \tag{2·2c}$$

The Casimir operator $\hat{\Gamma}_{ab_1b_2}$, which commutes with any generator, is expressed as

$$\hat{\Gamma}_{ab_1b_2}=(1/2)\cdot(\hat{S}_0^2+\hat{T}_0^2+\hat{R}_0^2)$$

$$+(3/4)[(\hat{S}_-^0\hat{S}_+^0+\hat{S}_+^0\hat{S}_-^0)/2-(\hat{T}_-^0\hat{T}_+^0+\hat{T}_+^0\hat{T}_-^0)/2-(\hat{R}_-\hat{R}_++\hat{R}_+\hat{R}_-)/2]$$
$$=\hat{K}^2-(3/2)\cdot\hat{K}\ , \tag{2·3}$$
$$\hat{K}=(\hat{b}_1\hat{b}_1^*+\hat{b}_2\hat{b}_2^*-\hat{a}^*\hat{a})/2\ . \tag{2·4}$$

The operator $\hat{K}$ commutes with any generator. We can see that $(\hat{T}_\pm^0, \hat{T}_0)$ and $(\hat{R}_\pm, \hat{R}_0)$ form the $su(1,1)$-algebras, respectively, and $(\hat{S}_\pm^0, \hat{S}_0)$ the $su(2)$-algebra. The operators $\hat{T}$, $\hat{S}$ and $\hat{R}$, which are defined in the following, commute with $(\hat{T}_\pm^0, \hat{T}_0)$, $(\hat{S}_\pm^0, \hat{S}_0)$ and $(\hat{R}_\pm, \hat{R}_0)$, respectively :

$$\hat{T}=(\hat{b}_1\hat{b}_1^*-\hat{a}^*\hat{a})/2\ ,$$
$$\hat{S}=(\hat{b}_1^*\hat{b}_1+\hat{b}_2^*\hat{b}_2)/2\ ,$$
$$\hat{R}=(\hat{b}_2\hat{b}_2^*-\hat{a}^*\hat{a})/2\ . \tag{2·5}$$

The operator $\hat{K}$ can be expressed in the form

$$\hat{K}=(\hat{T}+\hat{S}+\hat{R}+1)/2\ . \tag{2·6}$$

As were done in (I) and (II), our starting task is to introduce a coherent state in the present system. In (A), we gave two forms for the coherent states. By modifying one of the two, we described, in (B), the statistically mixed state for a boson system interacting with an external harmonic oscillator. In the present notation, the form is expressed as follows :

$$|c^0\rangle=\left(\sqrt{\Gamma_0}\right)^{-1}\exp\left(\gamma_1\hat{T}_+^0\right)\exp\left(\gamma_2\hat{S}_+^0\right)\exp\left(\delta\hat{b}_2^*\right)|0\rangle$$
$$=\left(\sqrt{\Gamma_0}\right)^{-1}\exp\left(\gamma_1\hat{a}^*\hat{b}_1^*\right)\exp\left(\gamma_2\hat{b}_1^*\hat{b}_2\right)\exp\left(\delta\hat{b}_2^*\right)|0\rangle\ , \tag{2·7a}$$
$$\Gamma_0=(1-|\gamma_1|^2)^{-1}\exp\left((1-|\gamma_1|^2+|\gamma_2|^2)|\delta|^2\cdot(1-|\gamma_1|^2)^{-1}\right)\ . \tag{2·7b}$$

Here, γ_1, γ_2 and δ denote complex parameters. In (A), we used V, w and v, which are related with the present in the form $\gamma_1=V(\sqrt{1+|V|^2})^{-1}$, $\gamma_2=w(v\sqrt{1+|V|^2})^{-1}$ and $\delta=v$. The normalization constant (2·7b) can be rewritten in the form

$$\Gamma_0=S(u,-v)e^{|\delta|^2}\ ,\qquad S(u,-v)=\frac{e^{v\frac{u}{1-u}}}{1-u}\ , \tag{2·8a}$$
$$u=|\gamma_1|^2\ ,\qquad v=|\gamma_1|^{-2}|\gamma_2|^2|\delta|^2\ . \tag{2·8b}$$

The function of $S(u,-v)$ is the generator of the Laguerre polynomials. The operators $\hat{T}_+^0$ and $\hat{S}_+^0$ play a role of generating the coherent state $|c^0\rangle$ on the state $|m\rangle=\exp(\delta\hat{b}_2^*)|0\rangle$, which obeys $\hat{T}_-^0|m\rangle=\hat{S}_-^0|m\rangle=0$. The state (2·7a) satisfies the following relations :

$$\hat{\gamma}_1^0|c^0\rangle=\gamma_1|c^0\rangle\ , \tag{2·9a}$$
$$\hat{\gamma}_2^0|c^0\rangle=\gamma_2(1-\epsilon(\hat{N}_{b_1}+\epsilon)^{-1})|c^0\rangle\ , \tag{2·9b}$$
$$\hat{\delta}^0|c^0\rangle=\delta|c^0\rangle\ . \tag{2·9c}$$

Here, $\hat{\gamma}_1^0$, $\hat{\gamma}_2^0$ and $\hat{\delta}^0$ are defined as

$$\hat{\gamma}_1^0 = (\hat{N}_{b_1}+1+\epsilon)^{-1}\hat{T}_-^0 \ , \tag{2·10a}$$
$$\hat{\gamma}_2^0 = \hat{S}_-^0(\hat{N}_{b_2}+1+\epsilon)^{-1}\left(1-\hat{N}_a(\hat{N}_{b_1}+\epsilon)^{-1}\right) \ , \tag{2·10b}$$
$$\hat{\delta}^0 = \hat{b}_2 \ , \tag{2·10c}$$

$$\hat{N}_a = \hat{a}^*\hat{a} \ , \qquad \hat{N}_{b_1} = \hat{b}_1^*\hat{b}_1 \ , \qquad \hat{N}_{b_2} = \hat{b}_2^*\hat{b}_2 \ . \tag{2·11}$$

The quantity ϵ is infinitesimal and $\epsilon(\hat{N}_{b_1}+\epsilon)^{-1}$ denotes a projection operator, the role of which is explained in (II). In the same meaning as that given in (II), the complex parameters γ_1, γ_2 and δ are nothing but the eigenvalues of $\hat{\gamma}_1^0$, $\hat{\gamma}_2^0$ and $\hat{\delta}^0$, respectively, and $|c^0\rangle$ is their eigenstate.

By regarding $|c^0\rangle$ as a trial state, the present variational method starts with the following relation :

$$\delta\int\langle c^0|i\partial_t - \hat{H}|c^0\rangle dt = 0 \ . \tag{2·12}$$

The state $|c^0\rangle$ satisfies the relation

$$\langle c^0|i\partial_t|c^0\rangle = (i/2)\cdot(x_1^*\dot{x}_1-\dot{x}_1^*x_1)+(i/2)\cdot(x_2^*\dot{x}_2-\dot{x}_2^*x_2)+(i/2)\cdot(y^*\dot{y}-\dot{y}^*y) \ . \tag{2·13}$$

Here, x_1, x_2 and y are defined as

$$\begin{aligned}
x_1 &= \gamma_1\sqrt{(\partial\Gamma_0/\partial|\gamma_1|^2)\cdot\Gamma_0^{-1}} \ , \qquad (|x_1|^2 = |\gamma_1|^2(\partial\Gamma_0/\partial|\gamma_1|^2)\cdot\Gamma_0^{-1})\\
x_2 &= \gamma_2\sqrt{(\partial\Gamma_0/\partial|\gamma_2|^2)\cdot\Gamma_0^{-1}} \ , \qquad (|x_2|^2 = |\gamma_2|^2(\partial\Gamma_0/\partial|\gamma_2|^2)\cdot\Gamma_0^{-1})\\
y &= \delta\sqrt{(\partial\Gamma_0/\partial|\delta|^2)\cdot\Gamma_0^{-1}} \ . \qquad (|y|^2 = |\delta|^2(\partial\Gamma_0/\partial|\delta|^2)\cdot\Gamma_0^{-1})
\end{aligned} \tag{2·14}$$

The relation (2·13) tells us that (x_1, x_1^*), (x_2, x_2^*) and (y, y^*) are boson-type canonical variables in classical mechanics. With the use of the relation (2·7b), we have

$$\begin{aligned}
x_1 &= \gamma_1\sqrt{1-|\gamma_1|^2+|\gamma_2|^2|\delta|^2}\cdot(1-|\gamma_1|^2)^{-1} \ ,\\
x_2 &= \gamma_2|\delta|\cdot\left(\sqrt{1-|\gamma_1|^2}\right)^{-1} \ ,\\
y &= \delta\sqrt{1-|\gamma_1|^2+|\gamma_2|^2}\cdot\left(\sqrt{1-|\gamma_1|^2}\right)^{-1} \ .
\end{aligned} \tag{2·15}$$

Inversely, the relation (2·15) gives us

$$\begin{aligned}
\gamma_1 &= x_1\cdot\left(\sqrt{1+|x_1|^2+|x_2|^2}\right)^{-1} \ ,\\
\gamma_2 &= x_2\cdot\left(\sqrt{1+|x_1|^2+|x_2|^2}\right)^{-1}\cdot\sqrt{1+|x_2|^2}\left(\sqrt{|y|^2-|x_2|^2}\right)^{-1} \ ,\\
\delta &= y\cdot|y|^{-1}\sqrt{|y|^2-|x_2|^2} \ .
\end{aligned} \tag{2·16}$$

In (A), we used the canonical variables (X, X^*), (Y, Y^*) and (Z, Z^*), which are related to

$$
\begin{aligned}
&X = y \cdot |y|^{-1}\sqrt{|y|^2 - |x_2|^2}\ (= v)\ , \\
&Y = x_1\ (= V\sqrt{1+|w|^2})\ , \\
&Z = x_2 y |y|^{-1}\ (= w)\ .
\end{aligned}
\qquad (2{\cdot}17)
$$

The expectation values of the generators for $|c^0\rangle$ are given in the following form :

$$
\begin{aligned}
&\langle c^0|\hat{T}^0_+|c^0\rangle = x_1^*\sqrt{1+|x_1|^2+|x_2|^2}\ (\ = \langle T^0_+\rangle\ )\ , \\
&\langle c^0|\hat{T}^0_-|c^0\rangle = x_1\sqrt{1+|x_1|^2+|x_2|^2}\ (\ = \langle T^0_-\rangle\ )\ , \\
&\langle c^0|\hat{T}_0|c^0\rangle = (1+2|x_1|^2+|x_2|^2)/2\ (\ = \langle T_0\rangle\ )\ , \qquad (2{\cdot}18a) \\
&\langle c^0|\hat{S}^0_+|c^0\rangle = x_2^*\sqrt{|y|^2-|x_2|^2}\sqrt{(1+|x_1|^2+|x_2|^2)\cdot(1+|x_2|^2)^{-1}}\ (\ = \langle S^0_+\rangle\ )\ , \\
&\langle c^0|\hat{S}^0_-|c^0\rangle = x_2\sqrt{|y|^2-|x_2|^2}\sqrt{(1+|x_1|^2+|x_2|^2)\cdot(1+|x_2|^2)^{-1}}\ (\ = \langle S^0_-\rangle\ )\ , \\
&\langle c^0|\hat{S}_0|c^0\rangle = (|x_1|^2+2|x_2|^2-|y|^2)/2\ (\ = \langle S_0\rangle\ )\ , \qquad (2{\cdot}18b) \\
&\langle c^0|\hat{R}_+|c^0\rangle = x_1^* x_2\sqrt{|y|^2-|x_2|^2}\cdot\left(\sqrt{1+|x_2|^2}\right)^{-1}\ (\ = \langle R_+\rangle\ )\ , \\
&\langle c^0|\hat{R}_-|c^0\rangle = x_1 x_2^*\sqrt{|y|^2-|x_2|^2}\cdot\left(\sqrt{1+|x_2|^2}\right)^{-1}\ (\ = \langle R_-\rangle\ )\ , \\
&\langle c^0|\hat{R}_0|c^0\rangle = (1+|x_1|^2-|x_2|^2+|y|^2)/2\ (\ = \langle R_0\rangle\ )\ . \qquad (2{\cdot}18c)
\end{aligned}
$$

The relation between the Poisson brackets for the above expectation values and the commutation relations for the generators was discussed in (A), and, then, we omit the discussion. The expectation value of $\hat{K}$ defined in the relation (2·4) is given by

$$
\langle c^0|\hat{K}|c^0\rangle = k = |y|^2/2 + 1\ . \qquad (2{\cdot}19)
$$

We define the classical correspondence of the Casimir operator $\hat{\Gamma}_{ab_1b_2}$ defined in the relation (2·3) in the form

$$
\begin{aligned}
\Gamma_{ab_1b_2} = &(1/2)\cdot(\langle S_0\rangle^2 + \langle T_0\rangle^2 + \langle R_0\rangle^2) \\
&+(3/4)\cdot(\langle S^0_+\rangle\langle S^0_-\rangle - \langle T^0_+\rangle\langle T^0_-\rangle - \langle R_+\rangle\langle R_-\rangle)\ .
\end{aligned}
\qquad (2{\cdot}20)
$$

With the use of the relations (2·18) and (2·19), $\Gamma_{ab_1b_2}$ is written as

$$
\Gamma_{ab_1b_2} = k^2 - (3/2)\cdot k + 3/4\ . \qquad (2{\cdot}21)
$$

The above should be compared with the relation (2·3). In this way, we obtain the classical counterpart of the $su(2,1)$-algebra expressed in terms of three kinds of boson operators.

Finally, we will sketch the relation between the set $(\hat{\gamma}_1^0, \hat{\gamma}_2^0, \hat{\delta})$ and the set $(\gamma_1, \gamma_2, \delta)$. The commutation relations for the above operators are given, for example, in the form

$$
\begin{aligned}
[\hat{\gamma}_1^0, \hat{\gamma}_1^{0*}] &= (\Delta_{\hat{N}_a}^{(+)} + \Delta_{\hat{N}_{b_1}}^{(+)} + \Delta_{\hat{N}_a}^{(+)}\Delta_{\hat{N}_{b_1}}^{(+)})(\hat{\gamma}_1^{0*}\hat{\gamma}_1^0) , \\
[\hat{\gamma}_2^0, \hat{\gamma}_2^{0*}] &= (\Delta_{\hat{N}_{b_1}}^{(+)} - \Delta_{\hat{N}_{b_2}}^{(-)} - \Delta_{\hat{N}_{b_1}}^{(+)}\Delta_{\hat{N}_{b_2}}^{(-)})(\hat{\gamma}_2^{0*}\hat{\gamma}_2^0) , \\
[\hat{\delta}^0, \hat{\delta}^{0*}] &= \Delta_{\hat{N}_{b_2}}^{(+)}(\hat{\delta}^{0*}\hat{\delta}^0) .
\end{aligned}
\tag{2·22}
$$

Here, $\Delta_{\hat{N}_c}^{(\pm)}$ for the operator $\hat{N}_c = \hat{c}^*\hat{c}$ $(\hat{c} = \hat{a},\ \hat{b}_1,\ \hat{b}_2)$ is defined as the difference for $F(\hat{N}_c)$:

$$
\Delta_{\hat{N}_c}^{(\pm)} F(\hat{N}_c) = \pm \left[F(\hat{N}_c \pm 1) - F(\hat{N}_c) \right] .
\tag{2·23}
$$

The other combinations are omitted. The corresponding Poisson brackets are given as

$$
\begin{aligned}
[\gamma_1, \gamma_1^*]_P &= (\partial_{N_a} + \partial_{N_{b_1}})|\gamma_1|^2 , \\
[\gamma_2, \gamma_2^*]_P &= (\partial_{N_{b_1}} - \partial_{N_{b_2}})|\gamma_2|^2 , \\
[\delta, \delta^*]_P &= \partial_{N_{b_2}}|\delta|^2 .
\end{aligned}
\tag{2·24}
$$

Here, the Poisson bracket is defined as

$$
\begin{aligned}
[A, B]_P = &(\partial_{x_1}A \cdot \partial_{x_1^*}B - \partial_{x_1^*}A \cdot \partial_{x_1}B) \\
&+(\partial_{x_2}A \cdot \partial_{x_2^*}B - \partial_{x_2^*}A \cdot \partial_{x_2}B) \\
&+(\partial_{y}A \cdot \partial_{y^*}B - \partial_{y^*}A \cdot \partial_{y}B) .
\end{aligned}
\tag{2·25}
$$

The quantities N_a, N_{b_1} and N_{b_2} denote the expectation values of $\hat{N}_a$, $\hat{N}_{b_1}$ and $\hat{N}_{b_2}$ for $|c^0\rangle$:

$$
N_a = |x_1|^2 , \qquad N_{b_1} = |x_1|^2 + |x_2|^2 , \qquad N_{b_2} = |y|^2 - |x_2|^2 .
\tag{2·26}
$$

Under the same argument as that given in (II), we can see that there exists the correspondence

$$
\hat{\gamma}_1^0 \sim \gamma_1 , \qquad \hat{\gamma}_2^0 \sim \gamma_2 , \qquad \hat{\delta}^0 \sim \delta .
\tag{2·27}
$$

This means that γ_1, γ_2 and δ introduced as the variational parameters are classical counterparts of the operators $\hat{\gamma}_1^0$, $\hat{\gamma}_2^0$ and $\hat{\delta}$, respectively.

§3. Deformation of the state $|c^0\rangle$

The most general form for the deformation of the state $|c^0\rangle$ given in the relation (2·7a) may be the following one :

$$
\begin{aligned}
|c\rangle = \left(\sqrt{\Gamma}\right)^{-1} &\exp\left(\gamma_1 \hat{a}^* \widetilde{f}(\hat{N}_a) \cdot \hat{b}_1^* \widetilde{g}_1(\hat{N}_{b_1})\right) \\
&\times \exp\left(\gamma_2 \hat{b}_1^* \widetilde{h}_1(\hat{N}_{b_1}) \cdot \widetilde{g}_2(\hat{N}_{b_2})\hat{b}_2\right) \\
&\times \exp\left(\delta \hat{b}_2^* \widetilde{h}_2(\hat{N}_{b_2})^{-1}\right) |0\rangle .
\end{aligned}
\tag{3·1}
$$

Properties of the function $\widetilde{f}(n)$, etc. are the similar to those presented in (II). The state $|c\rangle$ can be rewritten in the form

$$|c\rangle = \sqrt{\Gamma_0/\Gamma}\ \Omega(\hat{N}_a, \hat{N}_{b_1}, \hat{N}_{b_2})|c^0\rangle \ , \tag{3·2}$$

$$\Omega(\hat{N}_a, \hat{N}_{b_1}, \hat{N}_{b_2}) = f(\hat{N}_a)g_1(\hat{N}_{b_1})g_2(\hat{N}_{b_2})^{-1} \cdot h_1(\hat{N}_{b_1} - \hat{N}_a)g_1(\hat{N}_{b_1} - \hat{N}_a)^{-1}$$
$$\times g_2(\hat{N}_{b_1} + \hat{N}_{b_2} - \hat{N}_a)h_2(\hat{N}_{b_1} + \hat{N}_{b_2} - \hat{N}_a)^{-1} \ . \tag{3·3}$$

Here, $f(n)$, etc. is defined through the relation

$$\widetilde{f}(n) = f(n+1)f(n)^{-1} \ . \quad (n = 0, 1, 2, \cdots) \tag{3·4}$$

We described, in Ref.6), the statistically mixed state for a boson system interacting with an external harmonic oscillator. In this papers, we will investigate another case :

$$h_1 = g_1 \ , \qquad h_2 = g_2 \ , \tag{3·5}$$

i.e.,

$$\Omega(\hat{N}_a, \hat{N}_{b_1}, \hat{N}_{b_2}) = f(\hat{N}_a)g_1(\hat{N}_{b_1})g_2(\hat{N}_{b_2})^{-1} \ . \tag{3·6}$$

Then, the state (3·1) is reduced to

$$|c\rangle = \left(\sqrt{\Gamma}\right)^{-1} \exp\left(\gamma_1 \hat{a}^* \widetilde{f}(\hat{N}_a) \cdot \hat{b}_1^* \widetilde{g}_1(\hat{N}_{b_1})\right)$$
$$\times \exp\left(\gamma_2\ \delta\ \hat{b}_1^*\ \widetilde{g}_1(\hat{N}_{b_1})\right)$$
$$\times \exp\left(\delta\ \hat{b}_2^*\ \widetilde{g}_2(\hat{N}_{b_2})^{-1}\right)|0\rangle \ . \tag{3·7}$$

Here, it should be noted that we used the relation

$$\widetilde{g}_2(\hat{N}_{b_2})\hat{b}_2 \exp\left(\delta\ \hat{b}_2^*\ \widetilde{g}_2(\hat{N}_{b_2})^{-1}\right)|0\rangle = \delta \exp\left(\delta\ \hat{b}_2^*\ \widetilde{g}_2(\hat{N}_{b_2})^{-1}\right)|0\rangle \ . \tag{3·8}$$

The form (3·6) is a direct extension from that used in (II) for the case of two kinds of boson operators. The state (3·7) is expanded in the form

$$|c\rangle = \left(\sqrt{\Gamma}\right)^{-1} \sum_{n=0}^{\infty} \frac{f(n)\gamma_1^n}{n!}(\hat{a}^*)^n \sum_{m=0}^{\infty} \frac{g_1(n+m)(\gamma_2\delta)^m}{m!}(\hat{b}_1^*)^{(n+m)}$$
$$\times \sum_{l=0}^{\infty} \frac{g_2(l)^{-1}\delta^l}{l!}(\hat{b}_2^*)^l|0\rangle \ . \tag{3·9}$$

We impose the following condition :

$$f(0) = g_1(0) = g_2(0) = 1 \ . \tag{3·10}$$

The normalization constant $\mathit{\Gamma}$ is expressed as

$$\mathit{\Gamma} = \mathit{\Gamma}_\gamma \cdot \sum_{l=0}^{\infty} \frac{(|\delta|^2)^l}{l!} g_2(l)^{-2} \ . \tag{3·11}$$

As is shown in the relation (A·31), $\mathit{\Gamma}_\gamma$ is written as

$$\begin{aligned} \mathit{\Gamma}_\gamma &= \sum_{r=0}^{\infty} \frac{G_{1r}(0)}{r!} u^r \sum_{n=0}^{\infty} \sum_{s=0}^{n} (-)^{n-s} \frac{F_n(0) n!}{(s!)^2 (n-s)!} \\ &\quad \times \left(\frac{\partial}{\partial u} \right)^{r+s} \left[u^{n+s} S(u, -v) \right] \ , \qquad (3·12a) \\ f(n)^2 &= F(n) = \sum_{r=0}^{\infty} \frac{F_r(0)}{r!} n_r \ , \qquad (3·12b) \\ g_1(n)^2 &= G_1(n) = \sum_{r=0}^{\infty} \frac{G_{1r}(0)}{r!} n_r \ , \qquad (3·12c) \\ u &= |\gamma_1|^2 \ , \qquad v = |\gamma_1|^{-2} |\gamma_2|^2 |\delta|^2 \ . \qquad (3·12d) \end{aligned}$$

Here, $S(u, -v)$, u and v are defined in the relations (2·8a) and (2·8b), respectively. The meanings and the definitions of $G_{1r}(0)$, $F_r(0)$ and n_r are given in Appendix.

Next, we define the operators $\hat{\gamma}_1$, $\hat{\gamma}_2$ and $\hat{\delta}$ in the form

$$\begin{aligned} \hat{\gamma}_1 &= \mathit{\Omega}(\hat{N}_a, \hat{N}_{b_1}, \hat{N}_{b_2}) \ \hat{\gamma}_1^0 \ \mathit{\Omega}(\hat{N}_a, \hat{N}_{b_1}, \hat{N}_{b_2})^{-1} \ , \qquad (3·13a) \\ \hat{\gamma}_2 &= \mathit{\Omega}(\hat{N}_a, \hat{N}_{b_1}, \hat{N}_{b_2}) \ \hat{\gamma}_2^0 \ \mathit{\Omega}(\hat{N}_a, \hat{N}_{b_1}, \hat{N}_{b_2})^{-1} \ , \qquad (3·13b) \\ \hat{\delta} &= \mathit{\Omega}(\hat{N}_a, \hat{N}_{b_1}, \hat{N}_{b_2}) \ \hat{\delta}^0 \ \mathit{\Omega}(\hat{N}_a, \hat{N}_{b_1}, \hat{N}_{b_2})^{-1} \ . \qquad (3·13c) \end{aligned}$$

Here, $\hat{\gamma}_1^0$, $\hat{\gamma}_2^0$ and $\hat{\delta}^0$ are given in the relations (2·10a), (2·10b) and (2·10c), respectively. Since we have the relations (2·9a), (2·9b) and (2·9c), the following equations are derived :

$$\begin{aligned} \hat{\gamma}_1 |c\rangle &= \gamma_1 |c\rangle \ , \qquad (3·14a) \\ \hat{\gamma}_2 |c\rangle &= \gamma_2 (1 - \epsilon(\hat{N}_{b_1} + \epsilon)^{-1}) |c\rangle \ , \qquad (3·14b) \\ \hat{\delta} |c\rangle &= \delta |c\rangle \ . \qquad (3·14c) \end{aligned}$$

The commutation relations for the above operators are given, for example, in the form

$$\begin{aligned} [\hat{\gamma}_1, \hat{\gamma}_1^*] &= \left(\mathit{\Delta}_{\hat{N}_a}^{(+)} + \mathit{\Delta}_{\hat{N}_{b_1}}^{(+)} + \mathit{\Delta}_{\hat{N}_a}^{(+)} \mathit{\Delta}_{\hat{N}_{b_1}}^{(+)} \right) (\hat{\gamma}_1^* \hat{\gamma}_1) \ , \\ [\hat{\gamma}_2, \hat{\gamma}_2^*] &= \left(\mathit{\Delta}_{\hat{N}_{b_1}}^{(+)} - \mathit{\Delta}_{\hat{N}_{b_2}}^{(-)} - \mathit{\Delta}_{\hat{N}_{b_1}}^{(+)} \mathit{\Delta}_{\hat{N}_{b_2}}^{(-)} \right) (\hat{\gamma}_2^* \hat{\gamma}_2) \ , \\ [\hat{\delta}, \hat{\delta}^*] &= \mathit{\Delta}_{\hat{N}_{b_2}}^{(+)} (\hat{\delta}^* \hat{\delta}) \ . \qquad (3·15) \end{aligned}$$

The relation (3·15) is completely of the same form as that shown in the relation (2·22).

We can prove that $|c\rangle$ also satisfies the relation

$$\langle c|i\partial_t|c\rangle = (i/2)(x_1^*\dot{x}_1 - \dot{x}_1^* x_1) + (i/2)(x_2^*\dot{x}_2 - \dot{x}_2^* x_2) + (i/2)(y^*\dot{y} - \dot{y}^* y) \ . \tag{3·16}$$

Here, x_1, x_2 and y are defined as

$$x_1 = \gamma_1\sqrt{(\partial\varGamma/\partial|\gamma_1|^2)\cdot\varGamma^{-1}} \ , \qquad (|x_1|^2 = |\gamma_1|^2\cdot(\partial\varGamma/\partial|\gamma_1|^2)\cdot\varGamma^{-1}) \tag{3·17a}$$

$$x_2 = \gamma_2\sqrt{(\partial\varGamma/\partial|\gamma_2|^2)\cdot\varGamma^{-1}} \ , \qquad (|x_2|^2 = |\gamma_2|^2\cdot(\partial\varGamma/\partial|\gamma_2|^2)\cdot\varGamma^{-1}) \tag{3·17b}$$

$$y = \delta\sqrt{(\partial\varGamma/\partial|\delta|^2)\cdot\varGamma^{-1}} \ . \qquad (|y|^2 = |\delta|^2\cdot(\partial\varGamma/\partial|\delta|^2)\cdot\varGamma^{-1}) \tag{3·17c}$$

The relation (3·16) tells us that (x_1, x_1^*), (x_2, x_2^*) and (y, y^*) are boson-type canonical variables. In the case of $|c^0\rangle$, we can express γ_1, γ_2 and δ as functions of x_1, x_2 and y, which are shown in the relation (2·16). However, in the present case, it is in general impossible. The expectation values of $\hat{N}_a$, $\hat{N}_{b_1}$ and $\hat{N}_{b_2}$ for $|c\rangle$, which we denote N_a, N_{b_1} and N_{b_2}, are given as

$$\begin{aligned}
N_a &= |\gamma_1|^2\frac{\partial\varGamma}{\partial|\gamma_1|^2}\cdot\varGamma^{-1} = |x_1|^2 \ , \\
N_{b_1} &= |\gamma_1|^2\frac{\partial\varGamma}{\partial|\gamma_1|^2}\cdot\varGamma^{-1} + |\gamma_2|^2\frac{\partial\varGamma}{\partial|\gamma_2|^2}\cdot\varGamma^{-1} = |x_1|^2 + |x_2|^2 \ , \\
N_{b_2} &= |\delta|^2\frac{\partial\varGamma}{\partial|\delta|^2}\cdot\varGamma^{-1} - |\gamma_2|^2\frac{\partial\varGamma}{\partial|\gamma_2|^2}\cdot\varGamma^{-1} = |y|^2 - |x_2|^2 \ .
\end{aligned} \tag{3·18}$$

The form (3·18) is completely the same as that shown in the relation (2·26) for $|c^0\rangle$. Therefore, in the same logic adopted in (II), we can conclude that $\hat{\gamma}_1$, $\hat{\gamma}_2$ and $\hat{\delta}$ correspond to γ_1, γ_2 and δ, respectively.

We can calculate the expectation values of various operators for $|c\rangle$, the examples of which are shown in the relation (3·18). The other examples are the expectation values of the operators $\hat{\sigma}_+ = \hat{b}_1^*\,\widetilde{g}_1(\hat{N}_{b_1})\cdot\widetilde{g}_2(\hat{N}_{b_2})\hat{b}_2$ and $\hat{\tau}_- = \widetilde{g}_1(\hat{N}_{b_1})^{-1}\hat{b}_1\cdot\widetilde{f}(\hat{N}_a)^{-1}\hat{a}$. The straightforward calculation gives us

$$\hat{\sigma}_+|c\rangle = \gamma_2^*\frac{1}{|\gamma_2|^2}(\hat{N}_{b_1} - \hat{N}_a)|c\rangle \ , \tag{3·19a}$$

$$\hat{\tau}_-|c\rangle = \gamma_1(\hat{N}_{b_1} + 1)|c\rangle \ . \tag{3·19b}$$

Then, with the use of the relation (3·18), we have

$$\langle c|\hat{\sigma}_+|c\rangle = x_2^*\sqrt{\frac{\partial\varGamma}{\partial|\gamma_2|^2}\cdot\varGamma^{-1}} \ , \tag{3·20a}$$

$$\langle c|\hat{\tau}_-|c\rangle = x_1\sqrt{\left(\frac{\partial\varGamma}{\partial|\gamma_1|^2}\right)^{-1}\cdot\varGamma\cdot(1 + |x_1|^2 + |x_2|^2)} \ . \tag{3·20b}$$

The operator $\hat{\sigma}_+$ is nothing but $\hat{S}_+$ introduced in the next section and in §5, we investigate the form (3·20a).

§4. The $su(2,1)_q$-algebra in the present deformed boson scheme

It may be an interesting problem to investigate the $su(2,1)_q$-algebra in the present deformed boson scheme. If our understanding is correct, the algebra such as the $su(2,1)_q$-algebra has not been investigated by anyone. Our starting idea for formulating the $su(2,1)_q$-algebra is as follows : As was shown in the previous section, the state $|c\rangle$ is deformed from $|c^0\rangle$ through the three parts. As is clear from the state $|c^0\rangle$ shown in the relation (2·7a), the parts $\hat{T}_+^0$, $\hat{S}_+^0$ and $|m\rangle = \exp(\delta\hat{b}_2^*)|0\rangle$ are deformed. Therefore, for the deformation of the generators of the $su(2,1)$-algebra, $\hat{T}_\pm^0$ and $\hat{S}_\pm^0$ are deformed to $\hat{T}_\pm$ and $\hat{S}_\pm$ from the outside and the remaining generators $2\hat{T}_0$, $2\hat{S}_0$ and $\hat{R}_\pm$ are deformed through the commutation relation

$$[\hat{T}_+,\hat{T}_-] = -[2\hat{T}_0]_q \ , \qquad [\hat{S}_+,\hat{S}_-] = +[2\hat{S}_0]_q \ , \tag{4·1a}$$
$$[\hat{T}_\pm,\hat{S}_\mp] = \mp[\hat{R}_\pm]_q \ . \tag{4·1b}$$

The above is our starting idea for the deformation.

Following the basic form shown in the relation (II·4·3), we define the operators

$$\begin{aligned}
\hat{\alpha} &= \mathit{\Omega}(\hat{N}_a,\hat{N}_{b_1},\hat{N}_{b_2})^{-1}\ \hat{a}\ \mathit{\Omega}(\hat{N}_a,\hat{N}_{b_1},\hat{N}_{b_2}) \ ,\\
\hat{\beta}_1 &= \mathit{\Omega}(\hat{N}_a,\hat{N}_{b_1},\hat{N}_{b_2})^{-1}\ \hat{b}_1\ \mathit{\Omega}(\hat{N}_a,\hat{N}_{b_1},\hat{N}_{b_2}) \ ,\\
\hat{\beta}_2 &= \mathit{\Omega}(\hat{N}_a,\hat{N}_{b_1},\hat{N}_{b_2})\ \hat{b}_2\ \mathit{\Omega}(\hat{N}_a,\hat{N}_{b_1},\hat{N}_{b_2})^{-1} \ .
\end{aligned} \tag{4·2}$$

Here, $\mathit{\Omega}(\hat{N}_a,\hat{N}_{b_1},\hat{N}_{b_2})$ is given in the relation (3·6). Then, for $\hat{T}_\pm$ and $\hat{S}_\pm$, we give the forms

$$\hat{T}_+ = \hat{\alpha}^*\hat{\beta}_1^* \ , \qquad \hat{T}_- = \hat{\beta}_1\hat{\alpha} \ , \tag{4·3a}$$
$$\hat{S}_+ = \hat{\beta}_1^*\hat{\beta}_2 \ , \qquad \hat{S}_- = \hat{\beta}_2^*\hat{\beta}_1 \ . \tag{4·3b}$$

The explicit forms of $\hat{T}_\pm$ and $\hat{S}_\pm$ defined in the above are written as

$$\begin{aligned}
&\hat{T}_+ = f(\hat{N}_a)g_1(\hat{N}_{b_1})\ \hat{T}_+^0\ f(\hat{N}_a)^{-1}g_1(\hat{N}_{b_1})^{-1} \ , \qquad \hat{T}_+^0 = \hat{a}^*\hat{b}_1^* \ ,\\
&\hat{T}_- = f(\hat{N}_a)^{-1}g_1(\hat{N}_{b_1})^{-1}\ \hat{T}_-^0\ f(\hat{N}_a)g_1(\hat{N}_{b_1}) \ , \qquad \hat{T}_-^0 = \hat{b}_1\hat{a} \ ,
\end{aligned} \tag{4·4a}$$
$$\begin{aligned}
&\hat{S}_+ = g_1(\hat{N}_{b_1})g_2(\hat{N}_{b_2})^{-1}\ \hat{S}_+^0\ g_1(\hat{N}_{b_1})^{-1}g_2(\hat{N}_{b_2}) \ , \qquad \hat{S}_+^0 = \hat{b}_1^*\hat{b}_2 \ ,\\
&\hat{S}_- = g_1(\hat{N}_{b_1})^{-1}g_2(\hat{N}_{b_2})\ \hat{S}_-^0\ g_1(\hat{N}_{b_1})g_2(\hat{N}_{b_2})^{-1} \ , \qquad \hat{S}_-^0 = \hat{b}_2^*\hat{b}_1 \ .
\end{aligned} \tag{4·4b}$$

With the use of the relations (4·4a) and (4·4b), $[2\hat{T}_0]_q$, $[2\hat{S}_0]_q$ and $[\hat{R}_\pm]_q$ are obtained by the commutation relations (4·1a) and (4·1b) :

$$[2\hat{T}_0]_q = [\hat{N}_a+1]_f[\hat{N}_{b_1}+1]_{g_1} - [\hat{N}_a]_f[\hat{N}_{b_1}]_{g_1} \ , \tag{4·5a}$$
$$[2\hat{S}_0]_q = [\hat{N}_{b_1}]_{g_1}[\hat{N}_{b_2}+1]_{g_2} - [\hat{N}_{b_1}+1]_{g_1}[\hat{N}_{b_2}]_{g_2} \ , \tag{4·5b}$$

$$[\hat{R}_+]_q = \hat{\alpha}^*\hat{\beta}_2^*\left[[\hat{N}_{b_1}+1]_{g_1} - [\hat{N}_{b_1}]_{g_1}\right] \ ,$$
$$[\hat{R}_-]_q = \hat{\beta}_2\hat{\alpha}\left[[\hat{N}_{b_1}+1]_{g_1} - [\hat{N}_{b_1}]_{g_1}\right] \ . \qquad (4\cdot6)$$

Here, $[x]_f$, $[x]_{g_1}$ and $[x]_{g_2}$ are defined by

$$[x]_f = xf(x)^2f(x-1)^{-2} \ ,$$
$$[x]_{g_1} = xg_1(x)^2g_1(x-1)^{-2} \ , \qquad [x]_{g_2} = xg_2(x)^2g_2(x-1)^{-2} \ . \qquad (4\cdot7)$$

The operators $\hat{\alpha}$ and $\hat{\beta}_2$ are defined in the relation (4·2) :

$$\hat{\alpha} = f(\hat{N}_a+1)f(\hat{N}_a)^{-1}\ \hat{a} \ , \qquad \hat{\beta}_2 = g_2(\hat{N}_{b_2}+1)g_2(\hat{N}_{b_2})^{-1}\ \hat{b}_2 \ . \qquad (4\cdot8)$$

As was done in (II), let us introduce the operator $(\hat{E}_c, \hat{E}_c^*)$ for $c = a$, b_1 and b_2 defined as

$$\hat{E}_c = \left(\sqrt{\hat{N}_c+1}\right)^{-1}\hat{c} \ , \qquad \hat{E}_c^* = \hat{c}^*\left(\sqrt{\hat{N}_c+1}\right)^{-1} \ . \qquad (\hat{N}_c = \hat{c}^*\hat{c}) \qquad (4\cdot9)$$

The property is as follows :

$$\hat{E}_c\hat{E}_c^* = 1 \ , \qquad \hat{N}_c\hat{E}_c^*\hat{E}_c = \hat{E}_c^*\hat{E}_c\hat{N}_c = \hat{N}_c \ . \qquad (4\cdot10)$$

With the use of the operator $(\hat{E}_c, \hat{E}_c^*)$, the deformed generators in the present algebra can be expressed as

$$\hat{T}_+ = \hat{E}_a^*\hat{E}_{b_1}^*\sqrt{[\hat{T}_0-\hat{T}+1]_f[\hat{T}_0+\hat{T}]_{g_1}} = \sqrt{[\hat{T}_0-\hat{T}]_f[\hat{T}_0+\hat{T}-1]_{g_1}}\hat{E}_a^*\hat{E}_{b_1}^* \ ,$$
$$\hat{T}_- = \hat{E}_{b_1}\hat{E}_a\sqrt{[\hat{T}_0-\hat{T}]_f[\hat{T}_0+\hat{T}-1]_{g_1}} = \sqrt{[\hat{T}_0-\hat{T}+1]_f[\hat{T}_0+\hat{T}]_{g_1}}\hat{E}_{b_1}\hat{E}_a \ ,$$
$$[2\hat{T}_0]_q = [\hat{T}_0-\hat{T}+1]_f[\hat{T}_0+\hat{T}]_{g_1} - [\hat{T}_0-\hat{T}]_f[\hat{T}_0+\hat{T}-1]_{g_1} \ , \qquad (4\cdot11\text{a})$$
$$\hat{S}_+ = \hat{E}_{b_1}^*\hat{E}_{b_2}\sqrt{[\hat{S}+\hat{S}_0+1]_{g_1}[\hat{S}-\hat{S}_0]_{g_2}} = \sqrt{[\hat{S}+\hat{S}_0]_{g_1}[\hat{S}-\hat{S}_0+1]_{g_2}}\hat{E}_{b_1}^*\hat{E}_{b_2} \ ,$$
$$\hat{S}_- = \hat{E}_{b_2}^*\hat{E}_{b_1}\sqrt{[\hat{S}+\hat{S}_0]_{g_1}[\hat{S}-\hat{S}_0+1]_{g_2}} = \sqrt{[\hat{S}+\hat{S}_0+1]_{g_1}[\hat{S}-\hat{S}_0]_{g_2}}\hat{E}_{b_2}^*\hat{E}_{b_1} \ ,$$
$$[2\hat{S}_0]_q = [\hat{S}+\hat{S}_0]_{g_1}[\hat{S}-\hat{S}_0+1]_{g_2} - [\hat{S}+\hat{S}_0+1]_{g_1}[\hat{S}-\hat{S}_0]_{g_2} \ , \qquad (4\cdot11\text{b})$$
$$[\hat{R}_+]_q = \hat{E}_a^*\hat{E}_{b_2}^*\sqrt{[\hat{R}_0-\hat{R}+1]_f[\hat{R}_0+\hat{R}]_{g_2}}\left[[\hat{S}+\hat{S}_0+1]_{g_1} - [\hat{S}+\hat{S}_0]_{g_1}\right]$$
$$= \sqrt{[\hat{R}_0-\hat{R}]_f[\hat{R}_0+\hat{R}-1]_{g_2}}\left[[\hat{S}+\hat{S}_0+1]_{g_1} - [\hat{S}+\hat{S}_0]_{g_1}\right]\hat{E}_a^*\hat{E}_{b_2}^* \ ,$$
$$[\hat{R}_-]_q = \hat{E}_{b_2}\hat{E}_a\sqrt{[\hat{R}_0-\hat{R}]_f[\hat{R}_0+\hat{R}-1]_{g_2}}\left[[\hat{S}+\hat{S}_0+1]_{g_1} - [\hat{S}+\hat{S}_0]_{g_1}\right]$$
$$= \sqrt{[\hat{R}_0-\hat{R}+1]_f[\hat{R}_0+\hat{R}]_{g_2}}\left[[\hat{S}+\hat{S}_0+1]_{g_1} - [\hat{S}+\hat{S}_0]_{g_1}\right]\hat{E}_{b_2}\hat{E}_a \ . \qquad (4\cdot11\text{c})$$

With the use of various functions $f(x)$, $g_1(x)$ and $g_2(x)$, we are able to obtain various forms for the deformation in the case of the $su(2,1)$-algebra given in the framework of three kinds of boson operators. We will omit to give their explicit forms.

As was shown in the above, we performed the deformation of the $su(2,1)$-algebra by deforming $\hat{T}_{\pm}^{0}$ and $\hat{S}_{\pm}^{0}$ which are shown in the relation (4·4a) and (4·4b). It may be clear from the structure of $|c\rangle$ that these are regarded as the raising and the lowering operators for the minimum weight state constructed by $\hat{b}_2$-boson. In (A), we showed that in order to get the orthogonal sets, there exist eight possibilities for choosing the raising and the lowering operators together with the boson operators constructing the minimum weight state. Therefore, our present treatment can be applied to the other seven cases and we obtain various deformations. In the cases treated in (I) and (II), the deformation was uniquely performed.

§5. Discussions

The aim of this paper is to formulate, in the framework of the deformed boson scheme, the time-dependent variational method for many-body system constituted of three kinds of boson operators. The trial state $|c\rangle$ contains three complex parameters for the variation and to perform the variation, we have to calculate the expectation value of the Hamiltonian for the state $|c\rangle$. As can be seen in the relation (3·18) and (3·20), the expectation value of the Hamiltonian contains the normalization constant $\varGamma$ and its derivatives for the parameters. Therefore, it may be inevitable to investigate $\varGamma$ in more detail than that shown in the relations (3·11) and (3·12).

For the above-mentioned aim, let us investigate a rather simple example shown in the following :

$$\widetilde{f}(\hat{N}_a) = 1 \ , \qquad \widetilde{g}_1(\hat{N}_{b_1}) = \sqrt{1+\frac{\hat{N}_{b_1}}{2t}} \ , \qquad \widetilde{g}_2(\hat{N}_{b_2})^{-1} = \sqrt{1+\hat{N}_{b_2}} \ . \tag{5·1}$$

As is discussed later, t denotes a sufficiently large constant. The relation (5·1) gives us the forms

$$f(n)^2 = 1 \ , \tag{5·2a}$$

$$g_1(n)^2 = \left(1+\frac{n-1}{2t}\right)\cdot\left(1+\frac{n-2}{2t}\right)\cdots\left(1+\frac{1}{2t}\right)\cdot 1 \ , \tag{5·2b}$$

$$g_2(n)^{-2} = n! \tag{5·3}$$

The form (5·3) leads us to

$$\sum_{l=0}^{\infty}\frac{(|\delta|^2)^l}{l!}g_2(l)^{-2} = \sum_{l=0}^{\infty}(|\delta|^2)^l = \frac{1}{1-|\delta|^2} \ . \qquad (|\delta|^2<1) \tag{5·4}$$

Then, the normalization constant $\varGamma$ given in the relation (3·11) is written as

$$\varGamma = \varGamma_\gamma \cdot \frac{1}{1-|\delta|^2} \ . \qquad (5·5)$$

The relations (3·17a) and (3·17b) give us

$$u(\partial\varGamma_\gamma/\partial u)\cdot\varGamma_\gamma^{-1} = |x_1|^2+|x_2|^2 \ , \qquad (5·6a)$$
$$v(\partial\varGamma_\gamma/\partial v)\cdot\varGamma_\gamma^{-1} = |x_2|^2 \ . \qquad (5·6b)$$

The relation (3·17c) leads us to

$$|\delta|^2 = (|y|^2-|x_2|^2)\cdot(1+|y|^2-|x_2|^2)^{-1} \ . \qquad (5·7)$$

Here, u and v are defined in the relation (2·8b).

As was already mentioned, t is sufficiently large. Then, we take into account the effects of t^{-1} in the linear order :

$$g_1(n)^2 = 1+\frac{1}{4t}n(n-1) \ . \qquad (5·8)$$

In the case of $f(n)^2$ and $g_1(n)^2$ shown in the relations (5·2a) and (5·8), respectively, the relations (3·12b) and (3·12c) give us

$$F_0(0)=1 \ , \qquad F_r(0)=0 \quad \text{for} \quad r=1,2,3,\cdots, \qquad (5·9a)$$
$$G_{10}(0)=1 \ , \qquad G_{12}(0)=\frac{1}{2t} \ , \qquad G_{1r}(0)=0 \quad \text{for} \quad r=1,3,4,\cdots. \qquad (5·9b)$$

Then, $\varGamma_\gamma$ given in the relation (3·12a) can be written in the form

$$\varGamma_\gamma = \varGamma^0+\frac{1}{2t}\widetilde{\varGamma} \ , \qquad (5·10a)$$
$$\varGamma^0 = S(u,-v) \ , \qquad (5·10b)$$
$$\begin{aligned}\widetilde{\varGamma} &= \frac{1}{2}u^2\frac{\partial^2}{\partial u^2}S(u,-v) \\ &= S(u,-v)\frac{u^2}{(1-u)^4}\left[(1-u)^2+2(1-u)v+\frac{1}{2}v^2\right] \ . \end{aligned} \qquad (5·10c)$$

Here, $S(u,-v)$ is defined in the relation (2·8a). With the use of the relations (5·6a) and (5·6b), together with the relation (5·10a), we can determine u and v, i.e., $|\gamma_1|^2$ and $|\gamma_2|^2$, as functions of $|x_1|^2$, $|x_2|^2$ and $|y|^2$. In the framework of the linear order for t^{-1}, the relations (5·6a) and (5·6b) can be written in the following form :

$$u(\partial\varGamma^0/\partial u)\cdot(\varGamma^0)^{-1}+\frac{1}{2t}u\frac{\partial}{\partial u}(\widetilde{\varGamma}/\varGamma^0) = |x_1|^2+|x_2|^2 \ , \qquad (5·11a)$$
$$v(\partial\varGamma^0/\partial v)\cdot(\varGamma^0)^{-1}+\frac{1}{2t}v\frac{\partial}{\partial v}(\widetilde{\varGamma}/\varGamma^0) = |x_2|^2 \ . \qquad (5·11b)$$

In order to solve the relation (5·11) in the linear order for $1/(2t)$, we decompose u and v as follows :

$$u = u^0 + \frac{1}{2t}\widetilde{u} \ , \qquad v = v^0 + \frac{1}{2t}\widetilde{v} \ . \tag{5·12}$$

The parts u^0 and v^0 are determined by the relation

$$u^0 \left[(\partial\Gamma^0/\partial u)\cdot(\Gamma^0)^{-1}\right]^0 = |x_1|^2 + |x_2|^2 \ , \tag{5·13a}$$

$$v^0 \left[(\partial\Gamma^0/\partial v)\cdot(\Gamma^0)^{-1}\right]^0 = |x_2|^2 \ . \tag{5·13b}$$

Here, $[Z(u,v)]^0$ denotes the quantity Z at the point $u = u^0$ and $v = v^0$. With the use of the relation (5·10b), u^0 and v^0 are determined in the form

$$\begin{aligned} u^0 &= |x_1|^2\cdot(1+|x_1|^2+|x_2|^2)^{-1} \ , \\ v^0 &= |x_2|^2\cdot(1+|x_2|^2)\cdot|x_1|^{-2} \ . \end{aligned} \tag{5·14}$$

Then, the zero-th order for $|\gamma_1|^2$ and $|\gamma_2|^2$ is given as

$$\begin{aligned} \left[|\gamma_1|^2\right]^0 &= |x_1|^2\cdot(1+|x_1|^2+|x_2|^2)^{-1} \ , \\ \left[|\gamma_2|^2\right]^0 &= |x_2|^2\cdot(1+|x_2|^2)(1+|x_1|^2+|x_2|^2)^{-1} \\ &\quad \times(1+|y|^2-|x_2|^2)(|y|^2-|x_2|^2)^{-1} \ . \end{aligned} \tag{5·15}$$

Next, we investigate the method to determine $\widetilde{u}$ and $\widetilde{v}$. The right-hand sides of the relations (5·11a) and (5·11b) are of the zero-th order for $1/(2t)$. Therefore, the first order terms for $1/(2t)$ on the left-hand sides should vanish. From this idea, we obtain the following relations :

$$\Gamma_{uu}\widetilde{u} + \Gamma_{uv}\widetilde{v} = -B_u \ , \tag{5·16a}$$

$$\Gamma_{vu}\widetilde{u} + \Gamma_{vv}\widetilde{v} = -B_v \ . \tag{5·16b}$$

Here, Γ_{uu}, etc. are defined as

$$\begin{aligned} \Gamma_{uu} &= \left[\partial\Gamma^0/\partial u\cdot(\Gamma^0)^{-1} + u\frac{\partial}{\partial u}\left(\partial\Gamma^0/\partial u\cdot(\Gamma^0)^{-1}\right)\right]^0 \ , \\ \Gamma_{uv} &= \left[u\frac{\partial}{\partial v}\left(\partial\Gamma^0/\partial u\cdot(\Gamma^0)^{-1}\right)\right]^0 \ , \\ \Gamma_{vu} &= \left[v\frac{\partial}{\partial u}\left(\partial\Gamma^0/\partial v\cdot(\Gamma^0)^{-1}\right)\right]^0 \ , \\ \Gamma_{vv} &= \left[\partial\Gamma^0/\partial v\cdot(\Gamma^0)^{-1} + v\frac{\partial}{\partial v}\left(\partial\Gamma^0/\partial v\cdot(\Gamma^0)^{-1}\right)\right]^0 \ , \end{aligned} \tag{5·17}$$

$$B_u = \left[u\frac{\partial}{\partial u}\left(\widetilde{\Gamma}/\Gamma^0\right)\right]^0 \ , \qquad B_v = \left[v\frac{\partial}{\partial v}\left(\widetilde{\Gamma}/\Gamma^0\right)\right]^0 \ . \tag{5·18}$$

More explicitly, the above quantities are expressed as follows :

$$\begin{aligned}
&\Gamma_{uu} = (1-u^0)^{-2} + v^0(1+u^0)(1-u^0)^{-3} \ , \\
&\Gamma_{uv} = u^0(1-u^0)^{-2} \ , \\
&\Gamma_{vu} = v^0(1-u^0)^{-2} \ , \\
&\Gamma_{vv} = u^0(1-u^0)^{-1} \ , \qquad (5\cdot19) \\
&B_u = (u^0)^2\left[2(1-u^0)^2 + 2(2+u^0)(1-u^0)v^0 + (1+u^0)(v^0)^2\right](1-u^0)^{-5} \ , \\
&B_v = v^0(u^0)^2\left[2(1-u^0)+v^0\right](1-u^0)^{-4} \ . \qquad (5\cdot20)
\end{aligned}$$

With the help of the relation (5·16), together with the relations (5·19) and (5·20), we can determine $\widetilde{u}$ and $\widetilde{v}$ as functions of $|x_1|^2$, $|x_2|^2$ and $|y|^2$.

As was promised in §3, we investigate the form $\partial\Gamma/\partial|\gamma_2|^2\cdot\Gamma^{-1}$. This quantity can be expressed as

$$\begin{aligned}
\partial\Gamma/\partial|\gamma_2|^2\cdot\Gamma^{-1} &= \frac{|\delta|^2}{|\gamma_1|^2}\cdot\frac{1}{v}\cdot v\,(\partial\Gamma_\gamma/\partial v)\cdot\Gamma_\gamma^{-1} \\
&= |x_2|^2(|y|^2-|x_2|^2)(1+|y|^2-|x_2|^2)^{-1}(uv)^{-1} \ . \qquad (5\cdot21)
\end{aligned}$$

The part $(uv)^{-1}$ can be approximated in the form

$$(uv)^{-1} = (u^0v^0)^{-1}\left[1-\frac{1}{2t}(u^0v^0)^{-1}(v^0\widetilde{u}+u^0\widetilde{v})\right] \ . \qquad (5\cdot22)$$

The solutions obtained in the above lead us to the following results :

$$\begin{aligned}
&|x_2|^2(|y|^2-|x_2|^2)(1+|y|^2-|x_2|^2)^{-1}(u^0v^0)^{-1} \\
&\quad = (1+|x_1|^2+|x_2|^2)(1+|x_2|^2)^{-1}(|y|^2-|x_2|^2)(1+|y|^2-|x_2|^2)^{-1} \ , \qquad (5\cdot23) \\
&-(u^0v^0)^{-1}(v^0\widetilde{u}+u^0\widetilde{v}) \\
&\quad = (|x_1|^2+|x_2|^2)(1+|x_1|^2+|x_2|^2)(1+|x_2|^2)^{-1} \ . \qquad (5\cdot24)
\end{aligned}$$

Then, we have

$$\begin{aligned}
\partial\Gamma/\partial|\gamma_2|^2\cdot\Gamma^{-1} &= (1+|x_1|^2+|x_2|^2)(1+|x_2|^2)^{-1} \\
&\quad \times(|y|^2-|x_2|^2)(1+|y|^2-|x_2|^2)^{-1} \\
&\quad \times\left[1+\frac{1}{2t}(|x_1|^2+|x_2|^2)(1+|x_1|^2+|x_2|^2)(1+|x_2|^2)^{-1}\right] . \qquad (5\cdot25)
\end{aligned}$$

With the aid of the form (5·25), we can calculate the expectation value of $\hat{S}_+$ defined in the relation (4·4b) for $|c\rangle$, the explicit form of which is given in the relation (3·20) :

$$\langle c|\hat{S}_+|c\rangle = \langle c|\hat{b}_1^*\sqrt{1+\frac{\hat{N}_{b_1}}{2t}}\left(\sqrt{1+\hat{N}_{b_2}}\right)^{-1}\hat{b}_2|c\rangle$$

$$
\begin{aligned}
&= x_2^*\sqrt{1+|x_1|^2(1+|x_2|^2)^{-1}} \\
&\quad\times\frac{1}{\sqrt{2t}}\sqrt{2(t+|x_1|^2)+|x_2|^2-|x_1|^2(1-|x_1|^2)(1+|x_2|^2)^{-1}} \\
&\quad\times\sqrt{(|y|^2-|x_2|^2)(1+|y|^2-|x_2|^2)^{-1}}\ .
\end{aligned}
\tag{5·26}
$$

In (B), we have described a possible description of time-evolution of statistically mixed state for the following Hamiltonian expressed in terms of the present notations :

$$\hat{H} = \hat{K}_{b_1} + \hat{K}_{b_2} + \hat{V}_{b_1b_2}\ , \tag{5·27a}$$

$$\hat{K}_{b_1} = \omega\hat{N}_{b_1}\ , \tag{5·27b}$$

$$\hat{K}_{b_2} = \omega\hat{N}_{b_2}\ , \tag{5·27c}$$

$$\hat{V}_{b_1b_2} = -\gamma\sqrt{2t}\cdot i(\hat{S}_+ - \hat{S}_-)\ . \tag{5·27d}$$

Here, ω and γ are positive constants. Under the above Hamiltonian, we described the statistically mixed state of the system composed of the boson operator $(\hat{b}_1, \hat{b}_1^*)$. In this case, the system composed of the boson operator $(\hat{b}_2, \hat{b}_2^*)$ plays a role of the external environment. Further, we introduced the boson operator $(\hat{a}, \hat{a}^*)$ in order to describe the mixed state in terms of the phase space doubling. We reinvestigate roughly the Hamiltonian (5·27) in the present scheme. The expectation value of $\hat{H}$ for $|c\rangle$, H, is given in the following form :

$$H = 2\omega(k-1) + \omega\tau - \omega t - \gamma\cdot i(\tau_+^0 - \tau_-^0)\cdot\rho\sigma\ , \tag{5·28a}$$

$$k = |y|^2/2 + 1\ , \qquad (k = \langle c|\hat{K}|c\rangle) \tag{5·28b}$$

$$\tau = t + |x_1|^2\ , \qquad (|x_1|^2 = \langle c|\hat{N}_a|c\rangle) \tag{5·28c}$$

$$\tau_+^0 = x_2^*\sqrt{2\tau+|x_2|^2}\ , \qquad \tau_-^0 = x_2\sqrt{2\tau+|x_2|^2}\ , \tag{5·28d}$$

$$\rho = \sqrt{1+\frac{|x_1|^2}{1+|x_2|^2}}\sqrt{1-\frac{|x_1|^2}{1+|x_2|^2}\cdot\frac{1-|x_1|^2}{2\tau+|x_2|^2}}\ , \tag{5·28e}$$

$$\sigma = \sqrt{\frac{2(k-1)-|x_2|^2}{1+2(k-1)-|x_2|^2}}\ . \tag{5·28f}$$

The above is obtained by rewriting $\langle c|\hat{S}_+|c\rangle$ shown in the relation (5·26). We can see that the Hamiltonian (5·28a) is reduced to $H_{su(2,1)}$ in the relation (5·5) in (B) under the following approximation :

$$\rho \approx 1\ , \qquad \sigma \approx 1\ . \tag{5·29}$$

Of course, t, i.e., τ should be large. In (B), we investigated the time-evolution of the statistically mixed state for the Hamiltonian (5·28a) under the condition (5·29) and various results could be obtained. The condition $\rho \approx 1$ tells us that $|x_2|^2$ is larger than $|x_1|^2$.

Since $|x_1|^2 = \langle c|\hat{N}_a|c\rangle$ and $|x_1|^2 + |x_2|^2 = \langle c|\hat{N}_{b_1}|c\rangle$, the amplitude of the motion induced by the boson $(\hat{b}_1, \hat{b}_1^*)$ is larger than the measure of the statistical mixture. The condition $\sigma \approx 1$ implies, as was mentioned in (B), that the motion induced by the boson $(\hat{b}_1, \hat{b}_1^*)$ is of sufficiently long periodic motion.

Finally, we give some short comments. In (III), we presented the deformed boson scheme for the $su(2,1)$-algebra in three kinds of boson operators. Further, we showed that the description of the statistically mixed state given in (B) can be reproduced completely in the present scheme. In (I) and (II), rather well-known points are discussed systematically. But, Part (III) contains various aspects of the deformed boson scheme which are not so well known as those in (I) and (II). In this sense, it may be expected to investigate various problems in many-boson systems in the present scheme.

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Appendix A

—— *A method for calculating the normalization constant for the state* $|c\rangle$ ——

For the preparation of giving a method for calculating the normalization constant, we present two mathematical formulae. Let $R(n)$ denote a function for n $(= 0, 1, 2, \cdots)$, which obeys $|R(n)| < \infty$. The function $R(n)$ can be expressed in the following form :

$$R(n) = \sum_{r=0}^{n} \frac{R_r(0)}{r!} n_r \ . \tag{A·1}$$

Here, $R_r(0)$ and n_r are defined as

$$R_r(0) = \sum_{s=0}^{r} \frac{r!(-)^{r-s}}{s!(r-s)!} R(s) \ , \tag{A·2}$$

$$n_r = \begin{cases} 1 \ , & (r = 0) \\ n(n-1)\cdots(n-r+1) \ . & (r = 1, 2, 3, \cdots) \end{cases} \tag{A·3}$$

The proof is easy. If noting the relation $(1-1)^{n-s} = 1$ for $n = s$ and 0 for $n \neq s$, the straightforward calculation of the right-hand side of the relation (A·1) for the forms (A·2) and (A·3) leads us to the left-hand side of the relation (A·1). The meaning of $R_r(0)$ can be given in the following way : Let us define the function

$$R_r(k) = \sum_{s=0}^{r} \frac{r!(-)^{r-s}}{s!(r-s)!} R(s+k) \ . \qquad (k = 0, 1, 2, \cdots) \tag{A·4}$$

The case $k = 0$ is nothing but the form (A·2). For $R_r(k)$, we find the relation

$$\frac{R_{r-1}(k+1) - R_{r-1}(k)}{(k+1) - k} = R_r(k) \ . \tag{A·5}$$

The relation (A·5) tells us that $R_r(k)$ is the difference of r-th order at the point k. Therefore, $R_r(0)$ means the difference of r-th order at the point $k = 0$. The above is the meaning of $R_r(0)$. The quantity n_r defined in the relation (A·3) can be extended to the case $r = n+1,\ n+2, \cdots$:

$$n_r = n(n-1)\cdots(n-n+1)(n-(n+1)+1)\cdots(n-r+1) = 0 \ . \tag{A·6}$$

Therefore, the form (A·1) can be expressed as

$$R(n) = \sum_{r=0}^{\infty} \frac{R_r(0)}{r!} n_r \ . \tag{A·7}$$

Our next task is to prove the following relation :

$$\frac{(m+n)!}{m!} = \sum_{r=0}^{n} \frac{(n!)^2(-)^{n-r}}{(r!)^2(n-r)!} \frac{(m+n+r)!}{(m+n)!} \ . \qquad (m,\ n = 0,\ 1,\ 2, \cdots) \tag{A·8}$$

For the case $n = 0$, the relation (A·8) holds and, then, the cases $n = 1, 2, 3, \cdots$ are interesting. For this aim, we introduce two polynomials of degree n $(n = 1, 2, 3, \cdots)$ for the real variable x :

$$P_n(x) = (x+1)(x+2)\cdots(x+n) \ , \tag{A·9}$$

$$Q_n(x) = (-)^n n! + \sum_{r=1}^{n} \frac{(n!)^2(-)^{n-r}}{(r!)^2(n-r)!}(x+n+1)(x+n+2)\cdots(x+n+r) \ . \tag{A·10}$$

We calculate the values of $P_n(x)$ and $Q_n(x)$ at $(n+1)$ points of x given as

$$x_m = -(m+n+1) \ . \qquad (m = 0, 1, 2, \cdots, n) \tag{A·11}$$

The case of $P_n(x)$ is as follows :

$$P_n(x_m) = (-)^n \frac{(m+n)!}{m!} \ . \tag{A·12}$$

On the other hand, we have

$$\begin{aligned}Q_n(x_m) &= \sum_{r=0}^{m}(-)^{n-r}\frac{(n!)^2}{(r!)^2(n-r)!}(-)^r\frac{m!}{(m-r)!}\\ &= (-)^n n!\sum_{r=0}^{m}\frac{n!}{r!(n-r)!}\frac{m!}{r!(m-r)!}\\ &= (-)^n n!\cdot\frac{(m+n)!}{m!n!}\\ &= (-)^n\frac{(m+n)!}{m!}\ . \end{aligned} \tag{A·13}$$

Therefore, the following relation is obtained :

$$P_n(x_m) = Q_n(x_m)\ . \qquad (m=0,1,2,\cdots,n) \tag{A·14}$$

The above means that the values of the two polynomials of degree n, $P_n(x)$ and $Q_n(x)$, are equal to each other at the $(n+1)$ different points of x. Then, we have

$$P_n(x) = Q_n(x)\ . \tag{A·15}$$

Therefore, the relation (A·15) tells us that at $x=m$ $(m=0,1,2,\cdots)$, we get

$$P_n(m) = Q_n(m)\ , \tag{A·16a}$$

$$P_n(m) = (m+1)(m+2)\cdots(m+n) = \frac{(m+n)!}{m!}\ , \tag{A·16b}$$

$$\begin{aligned}Q_n(m) &= (-)^n n! + \sum_{r=1}^{n}\frac{(n!)^2(-)^{n-r}}{(r!)^2(n-r)!}(m+n+1)(m+n+2)\cdots(m+n+r)\\ &= \sum_{r=0}^{n}\frac{(n!)^2(-)^{n-r}}{(r!)^2(n-r)!}\cdot\frac{(m+n+r)!}{(m+n)!}\ . \end{aligned} \tag{A·16c}$$

Thus, we could prove the relation (A·8).

With the aid of the relations (A·7) (or (A·1)) and (A·8), we calculate the normalization constant Γ_γ for the state

$$|\gamma\rangle = \left(\sqrt{\Gamma_\gamma}\right)^{-1}\exp\left(\xi\hat{c}^*\widetilde{\phi}(\hat{N}_c)\cdot\hat{d}^*\widetilde{\psi}(\hat{N}_d)\right)\exp\left(\eta\hat{d}^*\widetilde{\psi}(\hat{N}_d)\right)|0\rangle\ . \tag{A·17}$$

Here, ξ and η denote complex parameters. The operators $(\hat{c},\hat{c}^*)$ and $(\hat{d},\hat{d}^*)$ are boson operators and $\hat{N}_c$ and $\hat{N}_d$ are boson number operators ($\hat{N}_c=\hat{c}^*\hat{c}$; $\hat{N}_d=\hat{d}^*\hat{d}$). The operators $\widetilde{\phi}(\hat{N}_c)$ and $\widetilde{\psi}(\hat{N}_d)$ denote functions of $\hat{N}_c$ and $\hat{N}_d$ with the properties suitable for our discussion in this paper. The state $|\gamma\rangle$ is rewritten in the form

$$\begin{aligned}|\gamma\rangle &= \left(\sqrt{\Gamma_\gamma}\right)^{-1}\exp\left((\xi\hat{c}^*\widetilde{\phi}(\hat{N}_c)+\eta)\cdot\hat{d}^*\widetilde{\psi}(\hat{N}_d)\right)|0\rangle\\ &= \left(\sqrt{\Gamma_\gamma}\right)^{-1}\sum_{n=0}^{\infty}\frac{\psi(n)}{n!}(\xi\hat{c}^*\widetilde{\phi}(\hat{N}_c)+\eta)^n(\hat{d}^*)^n|0\rangle\ . \end{aligned} \tag{A·18}$$

Here, $\psi(n)$ is defined as

$$\psi(n) = \begin{cases} 1\ , & (n=0) \\ \widetilde{\psi}(n-1)\widetilde{\psi}(n-2)\cdots\widetilde{\psi}(0)\ . & (n=1,2,3,\cdots) \end{cases} \tag{A·19}$$

Further, $|\gamma\rangle$ is rewritten as

$$|\gamma\rangle = \left(\sqrt{\mathit{\Gamma}_\gamma}\right)^{-1}\sum_{n=0}^{\infty}\frac{\psi(n)}{n!}\sum_{m=0}^{n}\frac{n!}{m!(n-m)!}\xi^m\eta^{n-m}\phi(m)(\hat{c}^*)^m(\hat{d}^*)^n|0\rangle\ . \tag{A·20}$$

Here, $\phi(m)$ is defined as

$$\phi(m) = \begin{cases} 1\ , & (m=0) \\ \widetilde{\phi}(m-1)\widetilde{\phi}(m-2)\cdots\widetilde{\phi}(0)\ . & (m=1,2,3,\cdots) \end{cases} \tag{A·21}$$

Then, $\mathit{\Gamma}_\gamma$ is expressed in the form

$$\mathit{\Gamma}_\gamma = \sum_{n=0}^{\infty}\sum_{m=0}^{n}\mathit{\Psi}(n)\mathit{\Phi}(m)\frac{n!}{m![(n-m)!]^2}u^n v^{n-m}\ . \tag{A·22a}$$

$$\mathit{\Psi}(n) = \psi(n)^2\ , \qquad \mathit{\Phi}(m) = \phi(m)^2\ . \tag{A·22b}$$

The quantities u and v are related to $|\xi|^2$ and $|\eta|^2$ in the form

$$|\xi|^2 = u\ , \qquad |\eta|^2 = uv\ . \tag{A·23}$$

We apply the relation (A·7) (or (A·1)) to the form (A·22b) :

$$\mathit{\Phi}(m) = \sum_{r=0}^{\infty}\frac{\mathit{\Phi}_r(0)}{r!}m_r = \sum_{r=0}^{m}\mathit{\Phi}_r(0)\frac{m!}{r!(m-r)!}\ . \tag{A·24}$$

Then, $\mathit{\Gamma}_\gamma$ given in Eq.(A·22a) is expressed as

$$\mathit{\Gamma}_\gamma = \sum_{n=0}^{\infty}\frac{\mathit{\Phi}_n(0)}{n!}u^n\sum_{m=0}^{\infty}u^m L_m(-v)\mathit{\Psi}(m+n)\frac{(m+n)!}{m!}\ . \tag{A·25}$$

The symbol $L_m(-v)$ denotes the Laguerre polynomial. Further, we apply the relation (A·7) to $\mathit{\Psi}(m+n)$ in Eq.(A·25) :

$$\mathit{\Psi}(m+n) = \sum_{r=0}^{\infty}\frac{\mathit{\Psi}_r(0)}{r!}(m+n)_r\ . \tag{A·26}$$

Then, $\mathit{\Gamma}_\gamma$ is, further, rewritten as

$$\mathit{\Gamma}_\gamma = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\frac{\mathit{\Phi}_n(0)}{n!}\frac{\mathit{\Psi}_r(0)}{r!}u^{n+m}L_m(-v)(m+n)_r\frac{(m+n)!}{m!}\ . \tag{A·27}$$

With the aid of the relation (A·8), we have

$$
\begin{aligned}
(m+n)_r \frac{(m+n)!}{m!} &= (m+n)_r \sum_{s=0}^{n} \frac{(n!)^2(-)^{n-s}}{(s!)^2(n-s)!} \cdot \frac{(m+n+s)!}{(m+n)!} \\
&= \sum_{s=0}^{n} \frac{(n!)^2(-)^{n-s}}{(s!)^2(n-s)!} \cdot \frac{(m+n+s)!}{(m+n-r)!} \\
&= \sum_{s=0}^{n} \frac{(n!)^2(-)^{n-s}}{(s!)^2(n-s)!} (m+n+s)(m+n+s-1)\cdots(m+n-r+1) \ .
\end{aligned}
\tag{A·28}
$$

The relation (A·28) gives us

$$
\begin{aligned}
\Gamma_\gamma = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{n} \frac{\Phi_n(0)}{n!}\frac{\Psi_r(0)}{r!}(-)^{n-s}\frac{(n!)^2}{(s!)^2(n-s)!} \\
\times(m+n+s)(m+n+s-1)\cdots(m+n-r+1)u^{m+n}L_m(-v) \ .
\end{aligned}
\tag{A·29}
$$

We note the relation

$$
(m+n+s)(m+n+s-1)\cdots(m+n-r+1)u^{m+n} = u^r \left[\left(\frac{\partial}{\partial u}\right)^{r+s} u^{m+n+s} \right] \ . \tag{A·30}
$$

Thus, we have

$$
\begin{aligned}
\Gamma_\gamma &= \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{n} \frac{\Phi_n(0)}{n!}\frac{\Psi_r(0)}{r!}(-)^{n-s}\frac{(n!)^2}{(s!)^2(n-s)!} \\
&\qquad \times u^r \left(\frac{\partial}{\partial u}\right)^{r+s} u^{n+s} \left(\sum_{m=0}^{\infty} u^m L_m(-v)\right) \\
&= \sum_{r=0}^{\infty} \frac{\Psi_r(0)}{r!} u^r \sum_{n=0}^{\infty}\sum_{s=0}^{n} (-)^{n-s} \frac{\Phi_n(0) n!}{(s!)^2(n-s)!} \\
&\qquad \times \left(\frac{\partial}{\partial u}\right)^{r+s} \left[u^{n+s} S(u,-v)\right] \ .
\end{aligned}
\tag{A·31}
$$

Here, we used the well-known formula for the generating function of the Laguerre polynomial :

$$
\sum_{n=0}^{\infty} L_n(x) t^n = S(t,x) \ , \qquad S(t,x) = \frac{e^{-x\frac{t}{1-t}}}{1-t} \ . \tag{A·32}
$$

We have another form for Γ_γ by applying the relation (A·7) to the following case :

$$
\Psi(m+n)(m+n)! = \sum_{r=0}^{\infty} \frac{\Xi_r(0)}{r!}(m+n)_r \ . \tag{A·33}
$$

Of course, $\Xi_r(0)$ denotes the difference with r-th degree for $\Psi(m+n)(m+n)!$. With the use of the relation (A·33), Γ_γ given in the relation (A·25) can be rewritten as

$$
\Gamma_\gamma = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty} \frac{\Phi_n(0)\Xi_r(0)}{n!m!r!}(m+n)\cdots(m+n-r+1)u^{m+n}L_m(-v) \ . \tag{A·34}
$$

We note the relation

$$(m+n)(m+n-1)\cdots(m+n-r+1)u^{m+n} = u^r \left(\frac{\partial}{\partial u}\right)^r u^{m+n} \ . \tag{A·35}$$

Then, $\varGamma_\gamma$ is expressed as

$$\begin{aligned} \varGamma_\gamma &= \sum_{n=0}^{\infty}\sum_{r=0}^{\infty} \frac{\varPhi_n(0)}{n!}\frac{\varXi_r(0)}{r!} u^r \left(\frac{\partial}{\partial u}\right)^r u^n \left(\sum_{m=0}^{\infty} \frac{u^m}{m!} L_m(-v)\right) \\ &= \sum_{r=0}^{\infty} \frac{\varXi_r(0)}{r!} u^r \left(\frac{\partial}{\partial u}\right)^r \left[e^u I_0(2\sqrt{uv}) \sum_{n=0}^{\infty} \frac{\varPhi_n(0)}{n!} u^n \right] \ . \end{aligned} \tag{A·36}$$

Here, $I_0(2\sqrt{uv})$ denotes the modified Bessel function with the 0-th degree :

$$I_0(2\sqrt{uv}) = J_0(2i\sqrt{uv}) = \sum_{n=0}^{\infty} \frac{(uv)^n}{(n!)^2} \ . \tag{A·37}$$

It should be noted that the notations used in this Appendix correspond to the following :

$$\begin{aligned} &\xi \longrightarrow \gamma_1 \ , \qquad \eta \longrightarrow \gamma_2\delta \ , \\ &u \longrightarrow |\gamma_1|^2 \ , \qquad v \longrightarrow |\gamma_1|^{-2}|\gamma_2|^2|\delta|^2 \ , \\ &(\hat{c}, \hat{c}^*) \longrightarrow (\hat{a}, \hat{a}^*) \ , \qquad (\hat{d}, \hat{d}^*) \longrightarrow (\hat{b}_1, \hat{b}_1^*) \ , \\ &\widetilde{\phi}(\hat{N}_c) \longrightarrow \widetilde{f}(\hat{N}_a) \ , \qquad \widetilde{\psi}(\hat{N}_d) \longrightarrow \widetilde{g}_1(\hat{N}_{b_1}) \ , \\ &\varPhi(n) \longrightarrow F(n) \ , \qquad \varPsi(n) \longrightarrow G_1(n) \ . \end{aligned} \tag{A·38}$$

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